MONOTONE INEQUALITIES ON ISING AND POTTS MODELS

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1. Introduction and Background

Consider a finite graph $G = (V, E)$, a configuration of the graph is a function $\sigma : V \to S$ which assigns each vertex $x \in V$ with a spin value $\sigma(x) \in S$, S is the state space. Then we define the Hamilton of the spin system

$$H(\sigma) = \sum_{x\sim y} U(\sigma(x), \sigma(y)) + \sum_{x} V(\sigma(x)).$$

with a symmetric function $U : S \times S \to \mathbb{R} \cup \{\infty\}$, the neighbor-interaction, and a self-energy function which can represent the influence of external field: $V : S \to \mathbb{R}$. And for each configuration, we can define the Gibbs measure on the graph.

$$\mu_g(\sigma) = \frac{1}{Z_g} \exp\{-\beta H(\sigma)\},$$

where Z_g is the normalizing constant known as the partition function and β is the inverse temperature.

In the ferromagnetic Ising model, each site $x \in V$ can take either of two spin values, +1 (”spin up”) and -1 (”spin down”), so that the state space is equal to $S = \{-1, +1\}$. The Hamiltonian is given by $U(\sigma(x), \sigma(y)) = -\sigma(x)\sigma(y)$ and $V(\sigma(x)) = -h(x)\sigma(x)$. The function $h : V \to \mathbb{R}$ describes an external field.

The antiferromagnet Ising model is defined similarly to the ferromagnetic model, except that $U(\sigma(x), \sigma(y))$ is chosen to be $+\sigma(x)\sigma(y)$ instead of $-\sigma(x)\sigma(y)$. This means that neighboring sites now prefer to take opposite spins.

A natural generalization of the Ising model is the Potts model, where each spin may take $q \geq 2$ (rather than only two) different values. The state space is $S = \{1, 2, \ldots, q\}$. The Potts model also have ferromagnetic and antiferromagnetic cases. They have different neighborhood interactions and the same external field interaction. In the ferromagnetic Potts model,

$$U(\sigma(x), \sigma(y)) = -\mathbf{1}_{\sigma(x)=\sigma(y)}, \quad V(\sigma(x)) = -h(x)\mathbf{1}_{\sigma(x)=1}.$$

In the antiferromagnetic Potts model,

$$U(\sigma(x), \sigma(y)) = \mathbf{1}_{\sigma(x)=\sigma(y)}, \quad V(\sigma(x)) = -h(x)\mathbf{1}_{\sigma(x)=1}.$$

In Potts model, sometimes we assume that the external field is zero. In this context, random cluster model is introduced to give another way to sample from the Potts model. Firstly, the random-cluster measure $\phi^G_{p,q}$ for G with parameters $p \in [0,1]$ and $q > 0$ is defined to be the probability measure on the configurations on edge set $\{0,1\}^E$ which to each $\eta \in \{0,1\}^E$ assigns probability

$$\phi^G_{p,q}(\eta) = \frac{1}{Z^G_{p,q}} \left\{ \prod_{e \in \mathcal{B}} p^{\eta(e)}(1-p)^{1-\eta(e)} \right\} q^{k(\eta)}$$

where $k(\eta)$ is the number of connected components in the subgraph of G containing all vertices but only open edges.

2. Literature Review: Correlation Inequalities

In this section we introduce several useful inequalities on Ising model, these inequalities helps us understanding the correlation relationship between different state. The most famous inequality is the FKG inequality.

2.1. FKG Inequality on Ising Model.

Theorem 2.1. *The original FKG can be defined at any distributive lattice and log-supermodular function. [The FKG inequality] Let V be a finite distributive lattice, and let $\mu : L \to \mathbb{R}^+$be a log-supermodular function. Then, for any two increasing functions $f, g : L \to \mathbb{R}^+$, we have*

$$\left(\sum_{x\in L}\mu(x)f(x)\right)\left(\sum_{x\in L}\mu(x)g(x)\right) \leq \left(\sum_{x\in L}\mu(x)f(x)g(x)\right)\left(\sum_{x\in L}\mu(x)\right) \tag{1}$$

On Ising model, the increasing functions is the observation function on the configuration space$f : V^S \to \mathbb{R}$ such that if $\sigma \leq \sigma'$, then $f(\sigma) \leq f(\sigma')$ For such two increasing functions f, g, FKG inequality prove that

$$\langle fg\rangle \geq \langle f\rangle\langle g\rangle.$$

To apply FKG inequality on Ising model, we first take $f(\sigma) = \mathbb{I}_{\{\sigma_u=1\}}$ and $g(\sigma) = \mathbb{I}_{\{\sigma_v=1\}}$, both of f, g are increasing functions. Then (1) shows that

$$\mathbb{P}(\sigma_u = 1)\mathbb{P}(\sigma_u = 1) \leq \mathbb{P}(\sigma_u = 1, \sigma_v = 1) \tag{2}$$

Moreover ,we can also apply FKG inequality to the expectation functions. We take $f(\sigma) = \langle\sigma_u\rangle$ and $g(\sigma) = \langle\sigma_v\rangle$ Then (1) shows that

$$\langle\sigma_u\rangle\langle\sigma_v\rangle \leq \langle\sigma_u\sigma_v\rangle \tag{3}$$

2.2. GKS Inequality. Another important inequality on Ising model is he Griffiths, Kelly, and Sherman (GKS) inequalities. Assume that the external field is homogeneous, which means that the g_u have the same sign for all u, say $h_u \geq 0$ for all u, denoted by $\mathbf{h}(u) \geq 0$, (similar results hold by symmetry when $h_u < 0$ for all u). Then the first GKS inequality

$$\langle \sigma_A \rangle \geq 0$$

hold, which implies that when the external field is positive, any set of the spin have positive expectation. Also the second GKS inequality is

$$\langle \sigma_A \sigma_B \rangle \geq \langle \sigma_A \rangle \langle \sigma_B \rangle , \quad \boldsymbol{h} \geq 0$$

, where $A, B \subset \Lambda$ and $\sigma_A = \prod_{i \in A} \sigma_i$. Note that when $|A| = |B| = 1$ GKS inequality is a special case of FKG inequality.

The GKS inequality can be generalized to Potts model. Here we denote $\sigma_A^a = \prod_{i \in A} \mathbb{I}(\sigma_i = a)$ for some $A \in V$ and $a \in S$, here $\mathbb{I}$ is the indicator function.

Theorem 2.2. *Under the conditions above, if A and B are two subsets of V, $a, b \in \{1, \ldots, q\}$ and $a \neq b$, then (i)*

$$\langle \sigma_A^a \sigma_B^a \rangle \geq \langle \sigma_A^a \rangle \langle \sigma_B^a \rangle$$

and (ii)

$$\left\langle \sigma_A^a \sigma_B^b \right\rangle \geq \langle \sigma_A^a \rangle \langle \sigma_B^b \rangle$$

2.3. **Application of the Monotone Inequality.** A very useful corollary of the monotone inequality is to deduce the monoicity of parameter in the model. Ferromagnetic Ising model has monotonicity of magnetism with regard to temperature and external field.

Theorem 2.3. *In the ferromagnetic Ising model with positive external field $h > 0$, and $0 \leq \beta_1 \leq \beta_2$, then*

$$\langle \sigma_o \rangle_{\beta_1, h} \leq \langle \sigma_o \rangle_{\beta_2, h}.$$

If $\beta > 0$, and $0 \leq h_1 \leq h_2$, then

$$\langle \sigma_o \rangle_{\beta, h_1} \leq \langle \sigma_o \rangle_{\beta, h_2}.$$

Proof.

$$
\begin{aligned}
&\frac{d\langle\sigma_o\rangle_{\beta,h}}{d\beta} = \frac{d}{d\beta}\frac{\sum_\sigma \sigma_o e^{-\beta H(\sigma)}}{\sum_\sigma e^{-\beta H(\sigma)}} \\
&= \frac{\sum_\sigma \sigma_o \left(\sum_{x\sim y}\sigma_x\sigma_y + \sum_x h(x)\sigma_x\right) e^{-\beta H(\sigma)}}{\left(\sum_\sigma e^{-\beta H(\sigma)}\right)^2} \\
&- \frac{\sum_\sigma \sigma_o e^{H(\beta,\sigma)} \sum_\sigma e^{H(\beta,\sigma)}\left(\sum_{x\sim y}\sigma_x\sigma_y + \sum h(x)\sigma_x\right)}{\left(\sum_\sigma e^{-\beta H(\sigma)}\right)^2} \\
&= \sum_{x\sim y}\langle\sigma_o\sigma_x\sigma_y\rangle_{\beta,h} + \sum_x \; h(x)\langle\sigma_x\sigma_o\rangle_{\beta,h} - \sum_{x\sim y}\langle\sigma_o\rangle_{\beta,h}\langle\sigma_x\sigma_y\rangle_{\beta,h} - \sum_x \; h(x)\langle\sigma_x\rangle_{\beta,h}\langle\sigma_o\rangle_{\beta,h} \\
&\geq 0 \quad \text{(by GKS inequality)}
\end{aligned}
$$

□

2.4. **GHS Inequality on Ising Model.** There exists also another extremely useful inequality due to Griffiths, Hurst, and Sherman [7] (GHS) which is restricted to ferromagnetic pair interactions.

$$u_3(i,j,k) \leqq 0, \qquad \text{for} \qquad \boldsymbol{h} \geq 0.$$

where

$$u_3(i,j,k) = \langle\sigma_i\sigma_j\sigma_k\rangle - \langle\sigma_i\rangle\langle\sigma_j\sigma_k\rangle - \langle\sigma_j\rangle\langle\sigma_i\sigma_k\rangle - \langle\sigma_k\rangle\langle\sigma_i\sigma_j\rangle + 2\langle\sigma_i\rangle\langle\sigma_j\rangle\langle\sigma_k\rangle.$$

The derivation of this inequality which is much more specialized than either the GKS or the FKG inequalities, is in many ways also more complicated involving combinatorial analysis.

2.5. **Simon's inequality on Ising model.** Theorem 1.1. Let $\langle\sigma_\alpha\sigma_\gamma\rangle$ denote the two point function of a spin 1/2 nearest neighbor (infinite volume, free boundary condition) Ising ferromagnet at at some fixed temperature. Fix α,γ and B, a set of spins whose removal breaks the lattice in such a way that α and γ lie in distinct components. Then:

$$\langle\sigma_\alpha\sigma_\gamma\rangle \leqq \sum_{\delta\in B}\langle\sigma_\alpha\sigma_\delta\rangle\langle\sigma_\delta\sigma_\gamma\rangle.$$

2.6. **External field correlation on Ising Model.** Here we consider the influence of the variation of the external field on the spins, each external field g determines a Gibbs measure μ_g. We will denote expectation with respect to μ_g by $\langle\cdot\rangle_g$.

Theorem 2.4. *In the ferromagnetic Ising model, let* $g : V \to [-\infty, \infty]$ *and* $h : V \to [0, \infty]$ *be such that* $\min\{|g_v|, h_v\} < \infty$ *for all* $v \in V$*. Then for any* $o \in V$*,*

$$\langle\sigma_o\rangle_{g+h} - \langle\sigma_o\rangle_{g-h} \leqslant \langle\sigma_o\rangle_h - \langle\sigma_o\rangle_{-h}. \tag{4}$$

As a first corollary of theorem, for the Ising model on finite graphs, spin-spin correlations are maximised when the external field vanishes. Let $g : V \to [-\infty, \infty]$. Then for any $u, v \in V$,

$$\langle\sigma_u\sigma_v\rangle_g - \langle\sigma_u\rangle_g \langle\sigma_v\rangle_g \leqslant \langle\sigma_u\sigma_v\rangle_0.$$

This inequality also deduce the exponential decay in the lattice with external field.

If we take $G = \Lambda_N := [-N, N]^d \cap \mathbb{Z}^d$,then

$$\langle\sigma_0\rangle_h^+ - \langle\sigma_0\rangle_h^- \leqslant \langle\sigma_0\rangle^+ - \langle\sigma_0\rangle^- \leqslant C_1(\beta)e^{-C_2(\beta)N},$$

Remark 2.5. The FKG, GKS and the new correlation inequality are all invalid on Potts model. For instance, we consider a simple graph with two vertices $V = \{u, v\}$, let $Q = \{1, 2, 3\}$, $h(u) = 1, h(v) = 0, g(u) = 0, g(v) = 1$, then

$$\mu_{g+h}(\sigma_u = 1) - \mu_{g-h}(\sigma_u = 1) = 0.7345,$$

$$\mu_h(\sigma_u = 1) - \mu_{-h}(\sigma_u = 1) = 0.7236.$$

Therefore the inequality (4) is false in antiferromagnetic Potts model. As a result, the monotonicity in the antiferromagnetic Potts model can dot be deduced directly from these inequalities, therefore it's more difficult to explore its properties.

However, for antiferromagnetic Ising model,Suppose that the external field is zero and that the underlying graph is bipartite; this means that $G = (V, E)$ can be partitioned into two sets V_{even} and V_{odd} such that sites in V_{even} only have edges to sites in V_{odd} , and vice versa. Clearly, $\mathbf{Z}^d$ is an example of a bipartite graph. In this situation, we can reduce the antiferromagnetic Ising model to the ferromagnetic case by a simple spin-flipping trick: The bijection $\sigma \leftrightarrow \tilde{\sigma}$ of Ω defined by

$$\tilde{\sigma}(x) = \begin{cases} \sigma(x) & \text{if } x \in \mathcal{L}_{\text{even}} , \\ -\sigma(x) & \text{if } x \in \mathcal{L}_{\text{odd}} \end{cases}$$

maps any Gibbs measure for the antiferromagnetic Ising model to a Gibbs measure for the ferromagnetic Ising model with the same parameters, and vice versa.

3. Inequality and Correlation on Potts Model

In this section, we aim to explore the correlations on antiferromagnetic Potts model on the d-ary tree. It's well known that FKG inequality does not hold in antiferromagnetic Potts model. Now there's few studies on the monotone inequality of Potts model.

3.1. Chain Case.

Chain Case. Firstly we study from a simple model when the graph is a chain, that is $V = \{v_1, v_2, \cdots, v_n\}$ and $E = \{v_1v_2, v_2v_3, \cdots, v_{n-1}v_n\}$.

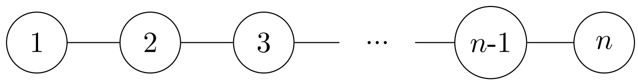


Figure 1. Potts model on a chain

We try to calculate the color distribution of v_n when the color of v_1 is given, i.e.

$$\mu_\beta\big(\sigma(v_n) = x|\sigma(v_1) = y\big), 1 \leq x, y \leq q.$$

where μ_β represents the Gibbs measure when the inverse temperature is at β. Denote $p = e^\beta$

Here we construct a Markov chain X_n on state space $\Omega = \{1, 2, \cdots, q\}$ with transition matrix P such that

$$P(x,y) = \begin{cases} \frac{p}{q-1+p} & \text{if } x = y \\ \frac{1}{q-1+p} & \text{if } x \neq y \end{cases}$$

Claim: Given the color of the first vertex $\sigma(v_1) = i$, the color distribution of $\sigma(v_n)$ in Potts model on a chain is equal to the distribution of X_n with starting point $X_1 = i$ and transition matrix P.

Proof. Firstly, we prove that

$$\mu_\beta\big(\sigma(v_i) = x|\sigma(v_{i+1}) = y\big) = P(x,y), \text{for all } 1 \leq i \leq n. \tag{5}$$

This is because when $x \neq y$

$$\begin{aligned}\frac{\mu_\beta\big(\sigma(v_i)=x|\sigma(v_{i+1})=y\big)}{\mu_\beta\big(\sigma(v_i)=y|\sigma(v_{i+1})=y\big)} &= \frac{\mu_\beta\big(\sigma(v_i)=x,\sigma(v_{i+1})=y\big)}{\mu_\beta\big(\sigma(v_i)=y,\sigma(v_{i+1})=y\big)} \\ &= \frac{\frac{1}{Z_\beta}\sum_{\sigma(v_i)=x,\sigma(v_{i+1})=y} p^{-H(\sigma)}}{\frac{1}{Z_\beta}\sum_{\sigma(v_i)=y,\sigma(v_{i+1})=y} p^{-H(\sigma)}} \\ &= \frac{1}{p}\end{aligned}$$

Also, since

$$\sum_{x\in\Omega}\mu_\beta\big(\sigma(v_i)=x|\sigma(v_{i+1})=y\big)=1$$

we know that (5) holds. Then using the property of conditional probability, we obtain that

$$\begin{aligned}\mu_\beta\big(\sigma(v_n)=x|\sigma(v_1)=y\big) = &\sum_{i_2,i_3,\cdots,i_{n-1}\in\Omega}\mu_\beta\big(\sigma(v_n)=x|\sigma(v_{n-1})=i_{n-1}\big) \\ &\mu_\beta\big(\sigma(v_{n-1})=i_{n-1}|\sigma(v_{n-2})=i_{n-2}\big)\cdots\mu_\beta\big(\sigma(v_2)=i_2|\sigma(v_1)=y\big) \\ &= \sum_{i_2,i_3,\cdots,i_{n-1}\in\Omega} P(i_{n-1},x)P(i_{n-2},i_{n-1})\cdots P(y,i_2) \\ &= P^{n-1}(y,x)=\mathbb{P}_y(X_n=x)\end{aligned}$$

□

Therefore we transform the question into calculating $P^n(x,y)$. Here we denote $u=(1,1,\cdots,1)$, then

$$P=\frac{1}{p+q-1}\big(u^Tu+(p-1)I_q\big)$$

and

$$P^{n-1}=\frac{1}{(p+q-1)^{n-1}}\big(Au^Tu+(p-1)^{n-1}I_q\big).$$

Since P^{n-1} is a stochastic matrix, we can calculate that

$$A=\frac{1}{q}[(p+q-1)^{n-1}-(p-1)^{n-1}].$$

As a result, we have

$$\mu_\beta(\sigma(n) = x|\sigma(1) = y) = \begin{cases} \frac{1}{q} - \frac{(p-1)^{n-1}}{(p+q-1)^{n-1}q} & \text{if } x \neq y \\ \frac{1}{q} + \frac{(p-1)^{n-1}(q-1)}{(p+q-1)^{n-1}q} & \text{if } x = y \end{cases}$$

This result shows that when n is odd, v_n is more likely to have the same color with v_1. However, when n is even, v_n is less likely to have the same color with v_1.

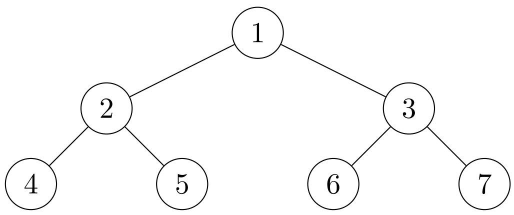


FIGURE 2. binary tree of depth-2

3.2. **Binary tree of depth-2.** We aim to find the distribution of v_1 when the colors of v_4, v_5, v_6, v_7 are given. Here we denote $\mu_{\beta,\xi}$ as the color distribution of v_1 given the boundary condition $\xi = (\xi_4, \xi_5, \xi_6, \xi_7)$,which means that $\sigma(v_4) = \xi_4, \sigma(v_5) = \xi_5, \sigma(v_6) = \xi_6, \sigma(v_7) = \xi_7$. Once the boundary condition is determined, we are interested in the probability of most-likely color, probability of least-likely color and the entropy of the distribution, respectively denoted by $S(\beta, \xi), I(\beta, \xi)$ and $H(\beta, \xi)$

$$S(\beta, \xi) = \max_i \mu_{\beta,\xi}(\sigma(v_1) = i)$$

$$I(\beta, \xi) = \min_i \mu_{\beta,\xi}(\sigma(v_1) = i)$$

$$H(\beta, \xi) = -\sum_i \mu_{\beta,\xi}(\sigma(v_1) = i) \log \mu_{\beta,\xi}(\sigma(v_1) = i)$$

Actually the property of the distribution is more complex than expected. we use two counterexample to illustrate this complexity.

Observation 1 The maximum of $S(\beta, \xi)$ may not be taken at $\xi_1 = (1, 1, 1, 1)$, that means for some β

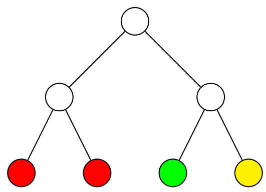
$$S(\beta, \xi_1) < \max_\xi S(\beta, \xi).$$

Justification 1 Take $q = 3$ and denote $\xi_2 = (1, 2, 1, 3)$, then

$$\lim_{\beta \to -\infty} S(\beta, \xi_1) = \frac{3}{4}$$

,while

$$\lim_{\beta \to -\infty} S(\beta, \xi_2) = 1$$



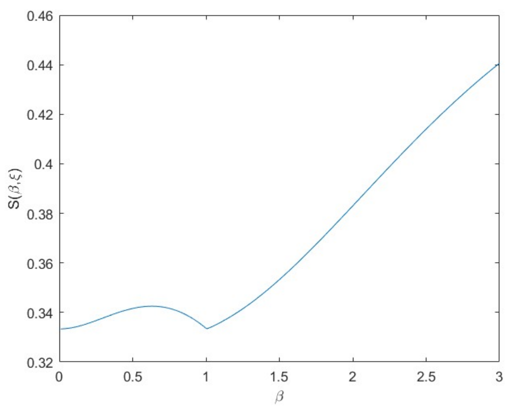


Figure 3. Relation between β and $S(\beta, \xi)$

Observation 2 Fix ξ, S, I, H may not increase with regard to β.

Justification 2 Actually we find that when $\xi = (1, 1, 2, 3)$, there exists $\beta < 0$ such that $\mu_{\beta,\xi}(\sigma(v_1) = i) = \mu_{0,\xi}(\sigma(v_1) = i)$ for all $i = 1, 2, 3$ as shown in figure 3. This excludes the possibility of constructing any possible monotone functions.

Hence we realize that the monotone property of this model is not clear. So we turn to study

$$S_m(\beta) = \sup_{\xi} S(\beta, \xi),$$
$$H_m(\beta) = \inf_{\xi} H(\beta, \xi).$$

We conjunct that S_m and H_m are both increasing with regard to β. Actually when the depth of the tree is small ($d = 2$ or $d = 3$), the conjecture can be verified directly by computer.

3.3. **case when $q = 2$ on $\mathbb{T}_2^n$.** Although the general monotone inequality is complicated in Potts model, the case when $q = 2$ is not so difficult. Here we consider the graph G as a 2-ary tree of depth n. Suppose that the two colors are $\{1, -1\}$. Also suppose we've already known the configuration on all leaves on the tree, denoted by ξ. Also denote $\langle \cdot \rangle_\mu$ as the expectation under the measure μ. Denote the nodes on $k-th$ level as $v_{k1}, v_{k2}, \cdots, v_{k2^k}$ and the root v_0. Let X_h be the average of 2^h values received at the node of level h. We claim that

Theorem 3.1.

$$\langle X_h \rangle_{\mu_{\beta,\xi}} = \frac{p^2-1}{p^2+1} \langle X_{h+1} \rangle_{\mu_{\beta,\xi}} \tag{6}$$

Specifically,

$$\langle \sigma_0 \rangle_{\mu_{\beta,\xi}} = \langle X_0 \rangle_{\mu_{\beta,\xi}} = (\frac{p^2-1}{p^2+1})^n \langle X_n \rangle_{\mu_{\beta,\xi}} = (\frac{p^2-1}{p^2+1})^n \frac{1}{2^n} \sum_{i=1}^{2^n} \xi_i \tag{7}$$

This implies that the color distribution of the root $\sigma(0)$ only depends on the expectation of the boundary condition, no matter how the configuration arranges. In fact, we can find some "conserved quantity" during the process of correlation.

Proof. Firstly we only consider the local correlation on each "unit branches".

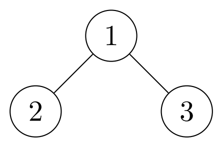


Figure 4. local correlation on unit branches

Suppose that we've already known the distribution of the color of nodes v_2, v_3, which are random variables with expectation e_2, e_3 respectively, v_2 has possibility p_2 to be 1 and

v_3 has possibility p_3 to be 1, obviously $e_2 = 2p_2 - 1, e_3 = 2p_3 - 1$. Therefore,

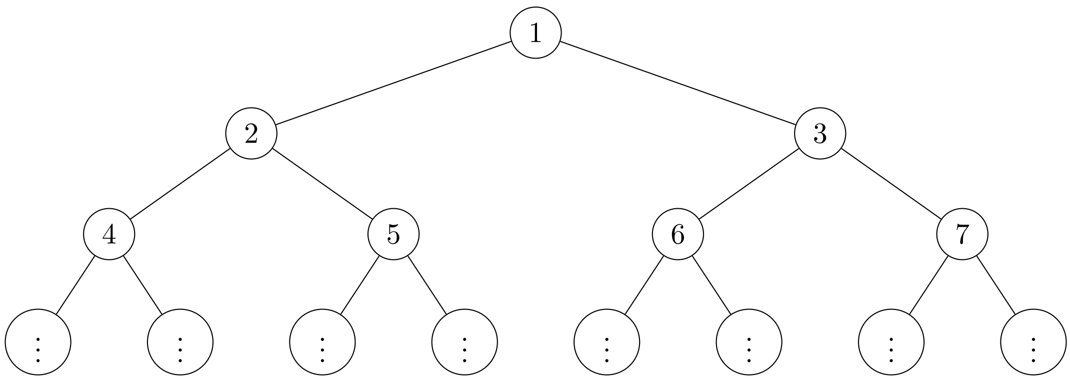
$$\mathbf{P}(\sigma(v_1 = 1)) = p_2p_3\frac{p^2}{1+p^2} + \frac{1}{2}(p_2(1-p_3) + p_3(1-p_2)) + (1-p_2)(1-p_3)\frac{1}{1+p^2} \quad (8)$$

$$= \frac{1}{1+p^2} + \frac{(p_2+p_3)(p^2-1)}{2(1+p^2)} \quad (9)$$

Then we can calculate that $\mathbf{E}(\sigma(v_1)) = \frac{(p_2+p_3-1)(p^2-1)}{p^2+1} = \frac{p^2-1}{2(p^2+1)}(e_2+e_3)$. Summing up all nodes at level h gives the formula.

Via this observation, we immediately know that the when $q = 2$, on the tree of arbitrary depth, the maximum of $S_m(\beta, \xi) = S(\beta, \xi_1) = \frac{1+\left(\frac{1-p^2}{p^2+1}\right)^n}{2}$, where ξ_1 represents the leaves are all known to be 1. Besides, the function $S(\beta, \xi)$ are monotone with regard to β, no matter the ξ is. $\square$

3.4. **Propagation Law.** In this section we consider a more general case, the critical case (q=3) on the $\mathbb{T}_2^n$.



Here we denote $\mu_{\beta,\xi,n}$ as the color distribution of v_1 given the boundary condition on all the leaves of the tree $\xi = (\xi_{2^n}, \xi_{2^n+1}, \cdots, \xi_{2^{n+1}-1})$.

Firstly we aim to prove the following "propagation law". Considering the sub-tree rooted at node 2 and 3, we can study their color distribution of them independently with the same boundary condition restricted to the sub-tree,i.e. $\mu_{\beta,\xi|_L,n-1}$ and $\mu_{\beta,\xi|_R,n-1}$. Here $\xi|_L$ represents $(\xi_{2^n}, \xi_{2^n+1}, \cdots, \xi_{2^n+2^{n-1}-1})$ and $\xi|_R$ represents $(\xi_{2^n+2^{n-1}}, \xi_{2^n+2^{n-1}+1}, \cdots, \xi_{2^n+2^{n-1}-1})$. For simplicity, let $\mu_1(x) = \mu_{\beta,\xi,n}(\sigma(v_1) = x), \mu_2(x) = \mu_{\beta,\xi|_L,n-1}(\sigma(v_1) = x)$ and $\mu_3 =$

$\mu_{\beta,\xi_R,n-1}(\sigma(v_3)=x)$ We claim that they have the following relationship:

$$\mu_1(i)=\frac{(1-(1-p)\mu_2(i))(1-(1-p)\mu_3(i))}{\sum_{i=1}^{q}(1-(1-p)\mu_2(i))(1-(1-p)\mu_3(i))} \tag{10}$$

Proof.

$$\begin{aligned}
\mu_1(x) &= \frac{1}{Z_1}\sum_{\sigma\in\Omega,\sigma(1)=x,\sigma|_B=\xi} p^{\#\{\{u,v\},u,v\in\mathbb{T},u\sim v,\sigma(u)=\sigma(v)\}} \\
&= \frac{1}{Z_1}\sum_{\sigma_L\in\Omega_L,\sigma_L|_B=\xi_L}\ \sum_{\sigma_R\in\Omega_R,\sigma_R|_B=\xi_R} p^{\#\{\{u,v\}\in\mathbb{T}_L,u\sim v,\sigma(u)=\sigma(v)\}+\mathbf{1}_{\sigma(v_2)=x}} \\
&\quad p^{\#\{\{u,v\}\in\mathbb{T}_R,u\sim v,\sigma(u)=\sigma(v)\}+\mathbf{1}_{\sigma(v_3)=x}} \\
&= \frac{1}{Z_1}\sum_{\sigma_L\in\Omega_L,\sigma_L|_B=\xi_L} p^{\#\{\{u,v\}\in\mathbb{T}_L,u\sim v,\sigma_L(u)=\sigma_L(v)\}+\mathbf{1}_{\sigma_L(v_1)=x}} \\
&\quad \sum_{\sigma_R\in\Omega_R,\sigma_R|_B=\xi_R} p^{\#\{\{u,v\}\in\mathbb{T}_R,u\sim v,\sigma_R(u)=\sigma_R(v)\}+\mathbf{1}_{\sigma_R(v_1)=x}} \\
&= \frac{1}{Z_1}(I_1\cdot I_2)
\end{aligned}$$

Here,

$$\begin{aligned}
I_1 &= \sum_{y\neq x}\ \sum_{\sigma_L\in\Omega_L,\sigma_L(1)=y,\sigma_L|_B=\xi_L} p^{\#\{\{u,v\}\in\mathbb{T}_L,u\sim v,\sigma_L(u)=\sigma_L(v)\}} \\
&\quad + p\sum_{\sigma_L\in\Omega_L,\sigma_L(1)=x,\sigma_L|_B=\xi_L} p^{\#\{\{u,v\}\in\mathbb{T}_L,u\sim v,\sigma_L(u)=\sigma_R(v)\}} \\
&= \sum_{y\neq x} Z_2\mu_2(y)+pZ_2\mu_2(x) \\
&= Z_2(1-(1-p)\mu_2(x))
\end{aligned}$$

Similarly, we can deduce that

$$I_2=Z_3(1-(1-p)\mu_3(x)).$$

Summing all the x we can deduce that (10). ☐

Then we deduce that

Proposition 3.2. *For $p > 1/4$,we have $\lim_{n\to\infty} \sup_\xi \mu_{\beta,\xi,n}(\cdot)$ tends to uniform distribution.*

Proof. The intuition of the proof is to construct a function acting on each distribution such that

$$\frac{1}{2}(f(\mu_2) + f(\mu_3)) < Cf(\mu_1), C < 1$$

for all possible distribution μ_2, μ_3, μ_1 with relationship as shown in (11). Here we choose $f(\mu) = \max\{\mu(1), \mu(2), \mu(3)\} - \min\{\mu(1), \mu(2), \mu(3)\}$. Then

$$f(\mu_1) = \frac{(1-(1-p)\mu_1(i))(1-(1-p)\mu_2(i)) - (1-(1-p)\mu_1(j))(1-(1-p)\mu_2(j))}{3 - 2c + c^2 \sum_{k=1}^{3} p_k q_k} \quad (11)$$

$$\leq \frac{1-p}{1+2p}(\mu_1(j) + \mu_2(j) - \mu_1(i) - \mu_2(i)) \quad (12)$$

$$\leq \frac{1-p}{1+2p}(f(\mu_1) + f(\mu_2)). \quad (13)$$

Take $C = \frac{2(1-p)}{1+2p}$. This implies that

$$f(\mu_{\beta,\xi,n}) \leq \frac{1}{2}C(f(\mu_{\beta,\xi|L,n-1}) + f(\mu_{\beta,\xi|R,n-1}))$$

After that, we can deduce by induction that for all ξ,

$$f(\mu_{\beta,\xi,n}) \leq C^n$$

Therefore,$\lim_{n\to\infty} \sup_\xi f(\mu_{\beta,\xi,n}) = 0$. This means that, as the depth of the tree increases, the distribution of the root point tends to be uniform distribution. □

The distribution $\mu^\xi_{n,\beta}(\sigma(v_1) = \cdot)$ can be described as a point $A = (x, y, z)$ on the simplex $S = \{(x, y, z) \in \mathbb{R}^3, x, y, z \geq 0, x + y + z = 1\}$. For any two distribution A, B on the simplex, define an operation $* : S \times S \to S$ by the law in (1), i,e.

$$A * B = (\frac{(1-(1-p)x_A)(1-(1-p)x_B)}{r}, \frac{(1-(1-p)y_A)(1-(1-p)y_B)}{r}, \frac{(1-(1-p)z_A)(1-(1-p)z_B)}{r})$$

where r is a normalizing constant.

The spreading of the distribution is a process like:

- Starting from 2^n points $A_0^1, A_0^2, \cdots, A_0^{2^n}$. The coordinate of each point is at $(1,0,0), (0,1,0)$ or $(0,0,1)$.
- At each time $t \geq 1$, we take $A_t^i = A_{t-1}^{2i-1} * A_{t-1}^{2i}$.
- We aim to prove: $\lim_{n\to\infty} A_n^1 = (1/3, 1/3, 1/3)$.

To describe the process of propagation, we aim to find some Lyapunov function $f : S \to \mathbb{R}$ such that for some $\epsilon > 0$ and arbitrary $A, B \in S$

- $(1/3, 1/3, 1/3)$ is the minimum of the function,
- $f(A * B) < \max(f(A), f(B))$,
- $\max_i f(A_k^i) < \max_i f(A_{k-1}^i)$.

When p is small (like the above-mentioned case $p < \frac{1}{4}$), finding such a Lyapunov function is easy. However, when p is close to 1, the inequality is very tight. The basic attempt, like $f(A) = \frac{\max\{x,y,z\}}{\min\{x,y,z\}}$,$f(A) = x^2 + y^2 + z^2$ and $f(A) = -(x \log x + y \log y + z \log z)$ is not true.

The one-step iteration is difficult, so we turn to the two-step iteration in the central area of the simplex.

Theorem 3.3 (Galanis-Goldberg-Yang, 2018)**.** *If we take* $f(A) = \frac{\max\{x_A, y_A, z_A\}}{\min\{x_A, y_A, z_A\}}$, *then*

(1) *For sufficient large* n, k, *their is* $\max_i f(A_k^i) \leq \frac{53}{27}$.

(2) *Under the condition of (1), the two-step recursion works. i.e. for* $k' > k$,$\max_i f(A_{k'}^i) < \max_i f(A_{k'-2}^i)$.

(3) *letting* $n \to \infty$, $f(A_n^1) \to 1$.

3.5. **Frozen Boundary.** In this part, our discussion is conducted under zero temperature conditions, which means that β approaches negative infinity.

First considering each "unity branches" as illustrated in Figure 4, we know that under the condition of $q = 3$, if the state of v_1 is given, for example, $v_1 = 1$, then v_2 and v_3 must be either (2, 3) or (3, 2). There are a total of $3^2 = 9$ possible distributions for v_2 and v_3. Therefore, we can say that among all distributions of the leaves, there are $\frac{2}{9}$ conditions that the entire system is in a frozen boundary which satisfies $v_1 = 1$.

More generally, the following conclusion holds.

Theorem 3.4. *Under the case when $q=3$, for $\forall i \in \{1,2,3\}, v_1 = i$. Among all distributions of the leaves, there are $\frac{2^{2^n-1}}{3^{2^n}}$ conditions that the entire system's state is in a frozen boundary which satisfies $v_1 = i$.*

Proof. Denote the number of layers as n. Assume that $v_1 = i$. When $n=2$, we have already shown that there are $\frac{2}{9}$ conditions that the entire system is in a frozen boundary.

Assuming the conclusion holds for the case of $n-1$ layers, we now prove its validity for the case of n layers.

Under the case of n layers, the tree has 2^n leaves. Therefore, there are 3^{2^n} possible distributions. Moreover, we know that the entire system is in a frozen boundary only if the entire system is in a frozen boundary after erasing the $n-th$ layer. Denote the number of frozen boundary conditions in the case of n layers as x_n, we have $\frac{x_n}{x_{n-1}} = 2^{2^{n-1}}$. Since $x_1 = 2$, we obtain that $x_n = 2^{2^n-1}$. Therefore, there are $\frac{2^{2^n-1}}{3^{2^n}}$ conditions that the entire system's state is in a frozen boundary which satisfies $v_1 = i$. □

3.6. **Infinite Tree Uniqueness.** In this section we let the depth of the tree tends to infinity. Let v_1 be the root of the tree. We call the Potts model has uniqueness on the infinite d-ary tree if, for all colours $c \in Q$, it holds that

$$\limsup_{n\to\infty} \max_{\xi:\partial\mathbb{T}_n^d \to Q} \left| \mu_{n,\beta}^{\xi}\left[\sigma\left(v_1\right)=c\right] - \frac{1}{q} \right| = 0.$$

It has non-uniqueness otherwise. The uniqueness of AF-Potts model tells about whether the boundary condition can affect the root when the depth of the tree tends to infinity. For the basic case when $d=2, q=3$, the infinity-uniqueness is true.

Theorem 3.5 (Galanis-Goldberg-Yang, 2018)**.** *When $d=2$,$q=3$, the 3 -state Potts model on the binary tree has uniqueness for all β.*

Also, if we've already known the model has uniqueness, then the speed of convergence if of order $O(n^{-1/2})$.

Theorem 3.6 (Gu-Wu-Yang, 2021)**.** *For the critical case $p_c = 1 - \frac{q}{d+1} > 0$, assume that the Potts model has uniqueness, then the convergence of marginal probability follows the* *power law that*

$$\lim_{n\to\infty} \frac{1}{n} \left(\max_{\xi:\partial\mathbb{T}_n^d \to Q} \left| \mu_{n,\beta}^{\xi} \left[\sigma(v_1) = 1\right] - \frac{1}{q} \right| \right)^{-2} = \frac{d^2-1}{6d^2} \left(\frac{q^2}{q-1} \right)^2 .$$

This theorem implies that, the distance between the distribution of the root and the uniform distribution, decays exponentially in distance. This is somehow a kind of "Weak Spatial Mixing".

Definition 3.7. (Weak Spatial Mixing (WSM)). Let $\mathcal{T}$ be a collection of rooted trees. The q-state Potts model on $\mathcal{T}$ exhibits weak spatial mixing (WSM) with exponential decay rate of $r \in (0,1)$, if there exists a constant $C > 0$, such that for any finite rooted tree $(T, v) \in \mathcal{T}$, any $\Lambda \subset V(T)\backslash\{v\}$, and any boundary condition $\tau : \Lambda \to [q]$, as well as any color $i \in [q]$, it holds that

$$\left| \mathbb{P}_{T,w}[\Phi(v) = i \mid \tau] - \frac{1}{q} \right| \leq C r^{\mathrm{dist}(v,\Lambda)},$$

where $\mathrm{dist}(v, \Lambda)$ denotes the graph distance from the root vertex v to the set Λ.

When considering two distinct boundary conditions, τ and τ', defined on the same vertex set Λ, we can compare the marginal probabilities of the root vertex v receiving color i for both boundary conditions. If the difference in marginal probabilities tends to zero as the distance increases, we speak of strong spatial mixing.

Definition 3.8. (Strong Spatial Mixing (SSM)). Let $\mathcal{T}$ be a collection of rooted trees. The q-state Potts model on $\mathcal{T}$ at parameter $w \geq 0$ exhibits strong spatial mixing (SSM) with exponential decay rate of $r \in (0,1)$, if there exists a constant $C > 0$, such that for any finite rooted tree $(T, v) \in \mathcal{T}$, any $\Lambda \subset V(T)\backslash\{v\}$, and any two boundary conditions $\tau, \tau' : \Lambda \to [q]$ differing on $\Delta_{\tau,\tau'} := \{u \in V(T) \mid \tau(u) \neq \tau'(u)\} \subset V(T)$, as well as any color $i \in [q]$, it holds that

$$\left| \mathbb{P}_{T,w}[\Phi(v) = i \mid \tau] - \mathbb{P}_{T,w}\left[\Phi(v) = i \mid \tau'\right] \right| \leq C r^{\mathrm{dist}\left(v, \Delta_{\tau,\tau'}\right)}.$$

Similarly, we can define the Strong Mixing time on Ising model.

Definition 3.9. The Ising measures $\mu_{V,\beta}^{\cdot,h}$ is said to satisfy the strong spatial mixing property in V with constants C and m (denoted by $SM(V,C,m)$), if for every $\Delta \subset V$ and $y \in \partial V$,

$$\sup_{\tau:\partial V \to \{\pm 1\}} \left\| \mu_{V,\beta}^{\tau,h}\Big|_{\Delta} - \mu_{V,\beta}^{\tau^y,h}\Big|_{\Delta} \right\|_{TV} \leqslant Ce^{-md(\Delta,y)},$$

where $\mu_{V,\beta}^{\tau,h}\Big|_{\Delta}$ denotes the marginal law of $(\sigma_v)_{v\in\Delta}$ under $\mu_{V,\beta}^{\tau}$, τ^y is obtained from τ by flipping the spin τ_y, and $\|\cdot\|_{TV}$ denotes total variation distance between measures.

As a corollary of Theorem 2.4, the new correlation inequality may contribute to prove the strong mixing property in Ising measure.

Corollary 3.10. *Let $d \geqslant 2$. Then for any $\beta \in [0,\beta_c)$, there exist $C, m \in (0,\infty)$ such that the Ising measures $\mu_{V,\beta}^{\cdot,h}$ satisfy the strong mixing property for all finite $V \subset \mathbb{Z}^d$ and $h : V \to \mathbb{R}$.*

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