Introduction of Ising and Potts Model

Let $G = (V, E)$ be a finite graph, a configuration of the graph is a function $\sigma : V \to Q$ which assigns each vertex $x \in V$ a spin value $\sigma(x) \in Q$, where Q is the state space.

In the **Ising model**, the state space $Q = \{+1, -1\}$. Each spin $x \in V$ can take either of two spin values, +1 ("spin up") and -1 ("spin down").

In the **Potts model**, the state space $Q = \{1, 2, \ldots, q\}$. Each spin may take $q \geq 3$ (rather than only two) different values.

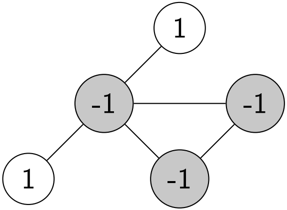

Figure: Ising configuration

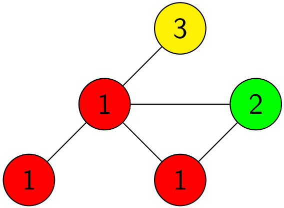

Figure: Potts configuration

Hamiltonian and Gibbs Measure

For a spin system, given a configuration σ, define its **Hamiltonian** as

$$H(\sigma) = \sum_{x \sim y} U(\sigma(x), \sigma(y)) + \sum_{x} V(\sigma(x)),$$

where $U : S \times S \to \mathbb{R}$ represents the neighbor-interaction of the spin system, and $V : S \to \mathbb{R} \cup \{\infty\}$ represents the influence of the external field.

For each configuration, define the **Gibbs measure** via

$$\mu_g(\sigma) = \frac{1}{Z_g} \exp\left\{-\beta H(\sigma)\right\},$$

where Z_g is the normalizing constant known as the partition function and β is the inverse temperature.

We call a model is **ferromagnetic** if two neighbor spins are more likely to be the same, and a model is **antiferromagnetic** if two neighbor spins are more likely to be different. This property is reflected in its Hamiltonian.

In the ferromagnetic Ising model, the Hamiltonian is

$$U(\sigma(x), \sigma(y)) = -\sigma(x)\sigma(y), \quad V(\sigma(x)) = -h(x)\sigma(x).$$

In the antiferromagnetic Ising model,

$$U(\sigma(x), \sigma(y)) = +\sigma(x)\sigma(y), \quad V(\sigma(x)) = -h(x)\sigma(x).$$

In the ferromagnetic Potts model,

$$U(\sigma(x), \sigma(y)) = -\mathbf{1}_{\sigma(x)=\sigma(y)}, \quad V(\sigma(x)) = -h(x)\mathbf{1}_{\sigma(x)=1}.$$

In the antiferromagnetic Potts model,

$$U(\sigma(x), \sigma(y)) = \mathbf{1}_{\sigma(x)=\sigma(y)}, \quad V(\sigma(x)) = -h(x)\mathbf{1}_{\sigma(x)=1}.$$

My Work

- Conducted an in-depth study of the history and literature in this field and completed a 20-page survey on the subject.
- Carefully read and verified every detail of two recent papers: A New Correlation Inequality for Ising Models with External Fields (Ding–Song–Sun, 2023) and Uniqueness for the 3-State Antiferromagnetic Potts Model on the Tree (Galanis–Goldberg–Yang, 2018).
- Assisted my advisor in reviewing and checking the details of a recent paper: Near-Optimal Bounds on the Antiferromagnetic Potts Model (Ferenc–Khallil–Guus, 2024).
- Explored the propagation law of the antiferromagnetic Potts model on binary trees.

Monotone Inequalities

In the ferromagnetic Ising model, many monotone inequalities are established to find out the correlation between different states. Denote expectation with respect to Gibbs measure μ by $\langle\cdot\rangle$. If there exists an external field g, denote the expectation as $\langle\cdot\rangle_g$.

Theorem (C. M. Fortuin-P. W. Kasteleyn-J. Ginibre 1971)

For arbitrary external field g, the Gibbs measure satisfies the following FKG inequality.

$$\langle\sigma_x\sigma_y\rangle_g \geq \langle\sigma_x\rangle_g\langle\sigma_y\rangle_g.$$

Theorem (Griffiths–Kelly-Sherman,1967)

Denote $\sigma_A = \prod_{i\in A}\sigma_i$, if the external field g is positive, i.e. $g(u) > 0$ for all $u \in V$ then

$$\langle\sigma_A\rangle_g \geq 0,$$

$$\langle\sigma_A\sigma_B\rangle_g \geq \langle\sigma_A\rangle_g\langle\sigma_B\rangle_g.$$

Application of Monotone Inequalities

Corollary: Ferromagnetic Ising model has monotonicity of magnetism with regard to temperature and external field.
In the ferromagnetic Ising model with positive external field $h > 0$, and $0 \leq \beta_1 \leq \beta_2$, then

$$\langle \sigma_o \rangle_{\beta_1, h} \leq \langle \sigma_o \rangle_{\beta_2, h}.$$

If $\beta > 0$, and $0 \leq h_1 \leq h_2$, then

$$\langle \sigma_o \rangle_{\beta, h_1} \leq \langle \sigma_o \rangle_{\beta, h_2}.$$

Remark: When $\beta = 0$, each node is in uniform distribution. While for $\beta = \infty$, in antiferromagnetic model, the configuration with positive probability must be q-coloring of the graph

Application of Monotone Inequalities

Proof.

$$\frac{d\langle\sigma_o\rangle_{\beta,h}}{d\beta} = \frac{d}{d\beta}\frac{\sum_\sigma \sigma_o e^{-\beta H(\sigma)}}{\sum_\sigma e^{-\beta H(\sigma)}}$$

$$= \frac{\sum_\sigma \sigma_o \left(\sum_{x\sim y}\sigma_x\sigma_y + \sum_x h(x)\sigma_x\right) e^{-\beta H(\sigma)}}{\left(\sum_\sigma e^{-\beta H(\sigma)}\right)^2}$$

$$- \frac{\sum_\sigma \sigma_o e^{H(\beta,\sigma)} \sum_\sigma e^{H(\beta,\sigma)} \left(\sum_{x\sim y}\sigma_x\sigma_y + \sum_x h(x)\sigma_x\right)}{\left(\sum_\sigma e^{-\beta H(\sigma)}\right)^2}$$

$$= \sum_{x\sim y}\langle\sigma_o\sigma_x\sigma_y\rangle_{\beta,h} + \sum_x \, h(x)\langle\sigma_x\sigma_o\rangle_{\beta,h}$$

$$- \sum_{x\sim y}\langle\sigma_o\rangle_{\beta,h}\langle\sigma_x\sigma_y\rangle_{\beta,h} - \sum_x \, h(x)\langle\sigma_x\rangle_{\beta,h}\langle\sigma_o\rangle_{\beta,h}$$

≥ 0 (by GKS inequality)

A new Correlation Inequality

Theorem (Ding-Song-Sun, 2023)

In ferromagnetic Ising model, let $g : V \to [-\infty, \infty]$ *and* $h : V \to [0, \infty]$ *be such that* $\min\{|g_v|, h_v\} < \infty$ *for all* $v \in V$. *Then for any* $o \in V$,

$$\langle \sigma_o \rangle_{g+h} - \langle \sigma_o \rangle_{g-h} \leqslant \langle \sigma_o \rangle_h - \langle \sigma_o \rangle_{-h}.$$

Corollary

$$0 \leq \langle \sigma_u \sigma_v \rangle_g - \langle \sigma_u \rangle_g \langle \sigma_v \rangle_g \leqslant \langle \sigma_u \sigma_v \rangle_0.$$

If we take $G = \Lambda_N := [-N, N]^d \cap \mathbb{Z}^d$,*then*

$$\langle \sigma_0 \rangle_h^+ - \langle \sigma_0 \rangle_h^- \leqslant \langle \sigma_0 \rangle^+ - \langle \sigma_0 \rangle^- \leqslant C_1(\beta) e^{-C_2(\beta)N},$$

Question: Can this inequality be generalized to Potts model ?

$$\mu_{g+h}(\sigma_o = 1) - \mu_{g-h}(\sigma_o = 1) \leq \mu_h(\sigma_o = 1) - \mu_{-h}(\sigma_o = 1)?$$

The answer is NO.

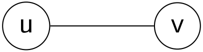

Figure: Counterexample1

Consider a simple graph with two vertices $V = \{u, v\}$, let $Q = \{1, 2, 3\}$, $h(u) = 1, h(v) = 0, g(u) = 0, g(v) = 1$, then

$$\mu_{g+h}(\sigma_u = 1) - \mu_{g-h}(\sigma_u = 1) = 0.7345,$$

$$\mu_h(\sigma_u = 1) - \mu_{-h}(\sigma_u = 1) = 0.7236.$$

AF-Potts Model on Tree

Next, we study the Gibbs measure of the antiferromagnetic (AF)-Potts model without external field on d-ary trees. Let $\mathbb{T}_n^d$ be the d-ary tree. We then use $\partial\mathbb{T}_n^d$ for the set of leaves. Denote $\mu_{n,\beta}^{\xi}$ as the Gibbs measure with temperature β, tree depth n and boundary condition $\partial\mathbb{T}_n^d = \xi$.

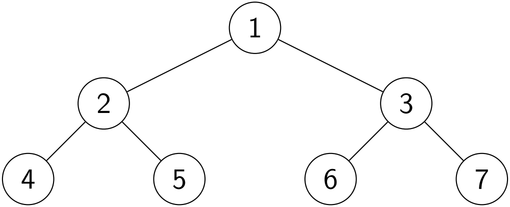

Figure: binary tree of depth-2

Infinite Tree Uniqueness

Definition

Let v_1 be the root of the tree. The Potts model has **uniqueness** on the infinite d-ary tree if, for all colours $c \in Q$, it holds that

$$\limsup_{n \to \infty} \max_{\xi : \partial \mathbb{T}_n^d \to Q} \left| \mu_{n,\beta}^{\xi} \left[\sigma \left(v_1 \right) = c \right] - \frac{1}{q} \right| = 0.$$

It has non-uniqueness otherwise.

Theorem (Galanis-Goldberg-Yang,2018)

When $d = 2$,$q = 3$, the 3 -state Potts model on the binary tree has uniqueness for all β.

Take $d = 2$ and $q = 3$. Consider the sub-tree rooted at two sons of v_1, we aim to find the relationship between the distribution of two layers. Let $p = e^{-\beta}$.

Question: Is there any relationship between $\mu^{\xi}_{n,\beta}$, $\mu^{\xi|_L}_{n-1,\beta}$ and $\mu^{\xi|R}_{n-1,\beta}$?

Here $\xi|_L$ represents $(\xi_{2^n}, \xi_{2^n+1}, \cdots, \xi_{2^n+2^{n-1}-1})$ and $\xi|_R$ represents $(\xi_{2^n+2^{n-1}}, \xi_{2^n+2^{n-1}+1}, \cdots, \xi_{2^n+2^{n-1}-1})$. For simplicity, let $\mu_1(x) = \mu^{\xi}_{n,\beta}(\sigma(v_1) = x), \mu_2(x) = \mu^{\xi|_L}_{n-1,\beta}(\sigma(v_1) = x)$ and $\mu_3(x) = \mu^{\xi|_R}_{n-1,\beta}(\sigma(v_1) = x)$. Then we have

$$\mu_1(x) = \frac{(1-(1-p)\mu_2(x))(1-(1-p)\mu_3(x))}{\sum_{i=1}^{3}(1-(1-p)\mu_2(i))(1-(1-p)\mu_3(i))}.$$

$$\mu_1(x) = \frac{(1-(1-p)\mu_2(x))(1-(1-p)\mu_3(x))}{\sum_{i=1}^{3}(1-(1-p)\mu_2(i))(1-(1-p)\mu_3(i))}.$$

The distribution $\mu_{n,\beta}^{\xi}(\sigma(v_1) = \cdot)$ can be described as a point $A = (x, y, z)$ on the simplex $S = \{(x, y, z) \in \mathbb{R}^3, x, y, z \geq 0, x + y + z = 1\}$. For any two distribution A, B on the simplex, define an operation $* : S \times S \to S$ by the law in (1), i,e.

$$A * B = (\frac{(1-(1-p)x_A)(1-(1-p)x_B)}{r},$$

$$\frac{(1-(1-p)y_A)(1-(1-p)y_B)}{r}, \frac{(1-(1-p)z_A)(1-(1-p)z_B)}{r}),$$

where r is a normalizing constant.

The spreading of the distribution is a process like:

- Starting from 2^n points $A_0^1, A_0^2, \cdots, A_0^{2^n}$. The coordinate of each point is at $(1,0,0),(0,1,0)$ or $(0,0,1)$.
- At each time $t \geq 1$, we take $A_t^i = A_{t-1}^{2i-1} * A_{t-1}^{2i}$.
- We aim to prove: $\lim_{n\to\infty} A_n^1 = (1/3, 1/3, 1/3)$.

Question: Is there any function $f : S \to \mathbb{R}$ such that for some $\epsilon > 0$ and arbitrary $A, B \in S$

- $(1/3, 1/3, 1/3)$ is the minimum of the function,
- $f(A * B) < \max(f(A), f(B))$?
- $\max_i f(A_k^i) < \max_i f(A_{k-1}^i)$?

Example : $f(A) = \frac{\max\{x,y,z\}}{\min\{x,y,z\}}$,$f(A) = x^2 + y^2 + z^2$ and $f(A) = -(x \log x + y \log y + z \log z)$ is not true.

Two-step Recursion

Theorem (Galanis-Goldberg-Yang, 2018)

If we take $f(A) = \frac{\max\{x_A, y_A, z_A\}}{\min\{x_A, y_A, z_A\}}$, *then*

1. *For sufficient large* n, k, *their is* $\max_i f(A_k^i) \leq \frac{53}{27}$.
2. *Under the condition of (1), the two-step recursion works. i.e. for* $k' > k$, $\max_i f(A_{k'}^i) < \max_i f(A_{k'-2}^i)$.
3. *letting* $n \to \infty$, $f(A_n^1) \to 1$.

Definition

The q-state Potts model exhibits strong spatial mixing (SSM) with exponential decay rate of $r \in (0,1)$, if there exists a constant $C > 0$, such that for any finite rooted tree (T, v), any $\Lambda \subset V(T) \backslash \{v\}$, and any two boundary conditions $\tau, \tau' : \Lambda \to [q]$ differing on $\Delta_{\tau,\tau'} := \{u \in \Lambda \mid \tau(u) \neq \tau'(u)\} \subset V(T)$, as well as any color $i \in [q]$, it holds that

$$\left|\mathbb{P}_{T;w}[\Phi(v) = i \mid \tau] - \mathbb{P}_{T;w}\left[\Phi(v) = i \mid \tau'\right]\right| \leq Cr^{\mathrm{dist}\left(v, \Delta_{\tau,\tau'}\right)}.$$

Theorem

Theorem 2. There exists a constant $K > 0$, such that for any $q \geq 3$, and any d such that $d + 1 \geq \frac{e-1/2}{e-1} q$, the q-state Potts model at parameter w on $\mathbb{T}_{d+1}$ exhibits SSM, provided

$$1 > w \geq 1 - \frac{q}{d+1}\left(1 - \frac{K}{d+1}\right).$$

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